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Ground-state magnetization curve of a generalized spin-1/2 ladder

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Employing a method of exact diagonalization for finite-size systems, we investigate the magnetization curve in the ground state of an antiferromagnetic spin-1/2 ladder with additional exchange interactions on diagonal bonds, which is equivalent to an antiferromagnetic spin-1/2 chain with bond-alternating nearest-neighbor and uniform next-nearest-neighbor interactions. It is found that a half-plateau appears in the magnetization curve in a certain range of the interaction constants. This result is discussed in connection with the necessary condition for the appearance of the plateau, recently given by Oshikawa et al.

Key Words: ground-state magnetization curve, generalized spin-1/2 ladder, spin-1/2 chain with competing interactions, half-plateau

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There has been a considerable current interest in the study of quantum spin systems with competing interactions, which exhibit a variety of fascinating phenomena originating from frustration and quantum fluctuation. In this paper we investigate the magnetization curve in the ground state of an antiferromagnetic spin-1/2 ladder with additional exchange interactions on diagonal bonds. We express the Hamiltonian describing this system in an external magnetic field as

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\rm Z}\,, \tag{1a}$$

$$\begin{aligned} \mathcal{H}_0 = (1+\alpha)\sum_{\ell=1}^{N/2} \vec{S}_{1,\ell}\cdot\vec{S}_{2,\ell} + J\sum_{j=1}^{2}\sum_{\ell=1}^{N/2} \vec{S}_{j,\ell}\cdot\vec{S}_{j,\ell+1} \\ + (1-\alpha)\sum_{\ell=1}^{N/2} \vec{S}_{1,\ell}\cdot\vec{S}_{2,\ell+1}\,, \end{aligned} \tag{1b}$$

$$\mathcal{H}_{\rm Z} = -H\sum_{j=1}^{2}\sum_{\ell=1}^{N/2} S^z_{j,\ell}\,, \tag{1c}$$

where $\vec{S}_{j,\ell}$ is the spin-1/2 operator at the jth leg and the ℓth rung; $1{+}\alpha$ and $1{-}\alpha$ $(0.0 \le \alpha \le 1.0)$ are, respectively, the interaction constants between neighboring spins along the rung and diagonal bonds; J $(J \geq 0.0)$ is the interaction constant between neighboring spins along the leg bond; H $(H \geq 0.0)$ is the magnitude, in an appropriate unit, of the magnetic field applied along the z-axis; N is the total number of spins in the system and is assumed to be a multiple of four. We impose periodic boundary conditions $(\vec{S}_{j,N+1} \equiv \vec{S}_{j,1})$. It should be noted that this system is equivalent to an antiferromagnetic spin-1/2 chain with both bond-alternating nearest-neighbor interactions, the interaction constants being $1{+}\alpha$ and $1{-}\alpha$, and uniform next-nearest-neighbor interactions, the interaction constant being J, which compete with each other.

The ground-state [1] and thermodynamic [2-4] properties of the system in the case of $\alpha=0.0$ and $H=0.0$ have been studied extensively. In particular, it has been found that as the value of J increases, the phase transition from the massless spin-fluid phase to the massive dimer phase occurs at $J=J_{\rm c}\sim 0.2412$ in the ground state [4-6]. As is well known, the exact ground-state magnetization curve in the case of $\alpha=J=0.0$ has been obtained by Griffiths [7]. The ground-state magnetization curves in the case of $\alpha=0.0$ [8] and in the case of $J=0.0$ [9] have also been numerically calculated for various values of J and α, respectively.

Obtaining the ground-state magnetization curve of the present system, we employ Sakai and Takahashi's method [10], by the use of which they have discussed the ground-state magnetization curve for a uniform spin-1 chain. The outline of this method, which is based on a method of exact diagonalization for finite-size systems combined with the conformal field theory, can be summarized as follows. Let $E_0(N,M)$ be the lowest-energy eigenvalue, within the subspace determined by the value $M=\sum_{j=1}^{2}\sum_{\ell=1}^{N/2} S_{j,\ell}^z(=0,\ 1,\ \cdots,\ N/2)$, of the Hamiltonian $\mathcal{H}_0$ for a given N. Then, the conformal field theory predicts that, if the state with the energy $E_0(N,M)$ is massless, the asymptotic behavior of $E_0(N,M)$ in the thermodynamic $(N\rightarrow\infty)$ limit has the form [11]

$$\frac{E_0(N,M)}{N}\sim\varepsilon(m)-C(m)\frac{1}{N^2}\qquad(N\rightarrow\infty)\tag{2}$$

with $m\equiv M/N$, where $\varepsilon(m)$ is the lowest energy per spin for a given m in the thermodynamic limit and $C(m)$ is a positive constant which is proportional to the product of the central charge and the sound velocity. Minimizing with respect to m the total energy, $\varepsilon_{\rm tot}=\varepsilon(m)-Hm$, per spin of the system described by the Hamiltonian $\mathcal{H}$, we can obtain the equation which relates

the ground-state magnetization $\langle m \rangle$ per spin in the thermodynamic limit with H as

$$\varepsilon'\big(\langle m \rangle\big) = H \ . \tag{3}$$

From Eq. (2) we can derive

$$\Delta E_0(N; M, M-1) \sim \varepsilon'(m_{-0}) - \frac{1}{2}\varepsilon''(m_{-0})\frac{1}{N} \qquad (N \to \infty) \ , \tag{4a}$$

$$\Delta E_0(N; M+1, M) \sim \varepsilon'(m_{+0}) + \frac{1}{2}\varepsilon''(m_{+0})\frac{1}{N} \qquad (N \to \infty) \ , \tag{4b}$$

where $\Delta E_0(N; M, M-1) \equiv E_0(N, M) - E_0(N, M-1)$. The essential point of Sakai and Takahashi's method [10] is that as far as the massless states are concerned, Eqs. (4a,b) hold when N is sufficiently large, and thus the value $H = \varepsilon'(m)$ [see Eq. (3)] of the magnetic field for a given m $(0 < m < 1/2)$ can be estimated by making an extrapolation which uses these equations to be $H = \varepsilon'(m_{-0}) = \varepsilon'(m_{+0})$. For the estimation of $H_{c0} \equiv \varepsilon'(0)$, which is the critical field at which $\langle m \rangle$ starts to increase from zero in the ground-state magnetization curve, we apply Shanks' transformation [12] to the sequences $\{\Delta E_0(N; 1, 0)\}$, following Sakai and Takahashi's procedure [13]. Furthermore, the saturation field $H_{\rm s} \equiv \varepsilon'(1/2)$ can be obtained analytically, since it is straightforward to diagonalize the Hamiltonian $\mathcal{H}_0$ within the $M = (N/2) - 1$ subspace; it is given by $H_{\rm s} = {\rm Max}(2,\ 1 + 2J + \alpha)$ in the present case. Recently, we have successfully applied the method to the case of an antiferromagnetic spin-1 chain with bond-alternating nearest-neighbor interactions and uniaxial single-ion-type anisotropy [14].

We obtain the ground-state magnetization curve in the thermodynamic limit for $J = 0.1$, 0.2, 0.3, and 0.4, for each of which various values of α are chosen. In the calculation we numerically diagonalize the Hamiltonian $\mathcal{H}_0$, using the computer program package KOBEPACK/S [15], to calculate $E_0(N, M)$ for $N = 8,\ 12,\ \cdots,\ 24$. Then, we can make the analysis for

$m=1/12$, $1/8$, $1/6$, $1/4$, $1/3$, $3/8$, and $5/12$. Our calculation shows that both $\Delta E_0(N;M+1,M)$ and $\Delta E_0(N;M,M-1)$ are almost linear functions of $1/N$ at least for $m=1/8$ and $3/8$ in accordance with the forms given by Eqs. (4a,b). The values of $\varepsilon'(m_{-0})$ and $\varepsilon'(m_{+0})$ for these m's can thus be estimated. In a similar way we estimate $\varepsilon'(m_{-0})$ and $\varepsilon'(m_{+0})$ for $m=1/12$, $1/6$, $1/3$, and $5/12$, assuming Eqs. (4a,b), although only two data are available. All the obtained results show that $\varepsilon'(m_{-0})$ and $\varepsilon'(m_{+0})$ coincide with each other within the numerical error. For $m=1/4$, on the other hand, Eqs. (4a, b) do or do not hold depending upon the values of α and J. The case where Eqs. (4a, b) do not hold is the case where the state with $m=1/4$ is massive and therefore $\varepsilon'(1/4_{-0})$ is smaller than $\varepsilon'(1/4_{+0})$. This means that, in the magnetization curve, there appears the half$\big(\langle m\rangle=1/4\big)$-plateau with the critical field $H_{\rm c1}\equiv\varepsilon'(1/4_{-0})$ at which the plateau starts and that $H_{\rm c2}\equiv\varepsilon'(1/4_{+0})$ at which it ends. We estimate the former and latter critical fields by applying Shanks' transformation [12] to the sequences $\{\Delta E_0(N;N/4,N/4-1)\}$ and $\{\Delta E_0(N;N/4+1,N/4)\}$, respectively.

We find that the half-plateau appears in the magnetization curve when $0.5\lesssim\alpha\lesssim0.95$ for $J=0.1$, when $0.2\lesssim\alpha\lesssim0.85$ for $J=0.2$, when $0.0\lesssim\alpha\lesssim0.8$ for $J=0.3$, and when $0.0<\alpha\lesssim0.75$ for $J=0.4$. As an example, we depict in Fig. 1 the magnetization curve with the half-plateau, obtained for $J=0.2$ and $\alpha=0.5$. Plotting versus α the critical fields $H_{\rm c0}$, $H_{\rm c1}$, and $H_{\rm c2}$ as well as the saturation field $H_{\rm s}$, we can draw the ground-state phase diagram on the H versus α plane; the result for $J=0.2$ is shown in Fig. 2.

We also calculate numerically the eigenfunctions of the lowest- and second-lowest-energy states within the $M=N/4$ subspace for the finite-N systems, and find that at least for a set of J and α giving the plateau,

the lowest-energy state for $M = N/4$ in the thermodynamic limit is doubly degenerate, one of the eigenfunctions of which has the periodicity $n = 4$ (in units of the lattice constant) concerning the translational symmetry. This result is consistent with the necessary condition $n\big(S - \langle m \rangle\big) =$ integer for the appearance of the plateau with the magnetization $\langle m \rangle$ (S is the magnitude of spins), which has recently been given by Oshikawa, Yamanaka, and Affleck [16]. Finally, it should be noted that very recently Totsuka [17] has clarified in an excellent way the mechanism for the appearance of the plateau in the present system, using a bosonization technique. According to this work, it becomes clear that the next-nearest-neighbor interaction plays a crucial role in the appearance of the plateau.

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Figure Captions

Fig. 1. Ground-state magnetization curve in the thermodynamic limit obtained for $J=0.2$ and $\alpha=0.5$. The closed circles show plots of $\langle m \rangle$ versus $H=\varepsilon'(\langle m \rangle)$. The solid lines are guides to the eye.

Fig. 2 Ground-state phase diagram on the H versus α plane in the thermodynamic limit for $J=0.2$. The closed circles show plots versus α of the critical fields $H_{\rm c0}$, $H_{\rm c1}$, and $H_{\rm c2}$ and also of the saturation field $H_{\rm s}$. The solid lines are guides to the eye. The magnetization $\langle m \rangle$ is given by $\langle m \rangle=0$, 1/4, and 1/2 in the regions A, B, and C, respectively. In the remaining region $\langle m \rangle$ increases continuously as H increases.